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PROJECTION CONDITIONS ON THE VORTICITY IN VISCOUS INCOMPRESSIBLE FLOWS

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SUMMARY

The problem of establishing appropriate conditions for the vorticity transport equation is considered. It is shown that, in viscous incompressible flows, the boundary conditions on the velocity imply conditions of an integral type on the vorticity. These conditions determine a projection of the vorticity field on the linear manifold of the harmonic vector fields. Some computational consequences of the above result in two-dimensional calculations by means of the nonprimitive variables, stream function and vorticity, are examined. As an example of the application of the discrete analogue of the projection conditions, numerical solutions of the driven cavity problem are reported.

KEY WORDS Vorticity Conditions Integral Conditions Incompressible Navier–Stokes Equations Computation of 2D Viscous Flows Vorticity/Stream Function Splitting Noniterative Algorithms

1. INTRODUCTION

In dealing with the vorticity/stream function equations it is common to attempt to assign boundary conditions to the vorticity equation as an equivalent substitute for the boundary conditions which are originally attached to the stream function equation. The problem of finding such vorticity conditions has been touched upon in many papers, especially in the computational literature, and a disconcerting variety of answers and working rules have been put forward (see, for instance, References 1 and 2 and the references therein). It is then worth attempting to elucidate the problem from a conceptual point of view.

In the present paper it is shown that the boundary conditions on the velocity can be transformed to completely equivalent conditions on the vorticity field. It must be remarked, however, that the latter are not of the usual, boundary-value type. As it turns out, the proper conditioning for the vorticity is of an integral (nonlocal) character and consists in fixing a projection of the vorticity field on the linear manifold of the harmonic vector fields.

This circumstance indicates that, once the problem has been stated in terms of boundary values for the stream function and its normal derivative, it will not admit an exact reformulation in terms of vorticity boundary values, except when the vorticity on the boundary is specified from the very outset, as, for instance, on planes of symmetry. As a practical consequence, in numerical calculations it is impossible to devise a boundary vorticity formula which will prove effective in the generality of cases. On the other hand, the integral projection conditions allow a splitting of the biharmonic equation into two second-order equations to be solved in sequence. A similar, yet incomplete, splitting is also possible for the general case of nonlinear equations in two and three dimensions.

The present paper is structured as follows. In Section 2 the projection conditions are derived in the simple case of the biharmonic equation, viz. the equation for steady creeping

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flows, where the scalar character of the problem makes the argument transparent. In Section 3 the same result is extended to the nonlinear time-dependent equations of incompressible viscous flows in two dimensions. The generalization to the three-dimensional case is presented in Section 4 by introducing scalar and vector potentials of the velocity field. Section 5 deals with the axisymmetric case. Section 6 describes the discrete analogue of the vorticity conditions for two-dimensional problems, and presents some computational schemes for the steady and unsteady equations. Finally, in Section 7, the simple numerical example of the driven cavity is considered, and finite difference solutions of this model problem are reported. The last section is devoted to a few conclusive remarks.

2. PROJECTION CONDITIONS

Let V be a simply connected bounded domain of the plane, with boundary S. Consider the biharmonic problem (Stokes problem)

$$\nabla^4\psi = f, \tag{1}$$

$$\psi\,|_S = a, \tag{2}$$

$$\partial\psi/\partial n\,|_S = b, \tag{3}$$

where f is a function given in V, and a and b are functions prescribed on S. Let the variable $\zeta = \nabla^2\psi$ be introduced, so that the fourth-order equation (1) for ψ can be rewritten as a system of two second-order equations for ζ and ψ, namely,

$$\nabla^2\zeta = f, \tag{4}$$

$$\nabla^2\psi = \zeta. \tag{5}$$

To split (4) from (5), conditions for ζ are required which are the exact substitutes for either (2) or (3) or both. To this end, notice that, by (5), (2) and (3), ζ is the Laplacian of some function ψ satisfying $\psi\,|_S = a$ and $\partial\psi/\partial n\,|_S = b$. This characterization of ζ can be translated into an equivalent one that depends only on the boundary data a and b of ψ, by virtue of the following basic remarks:

A function ζ in V is such that $\zeta = \nabla^2\psi$, with $\psi\,|_S = a$ and $\partial\psi/\partial n\,|_S = b$, if and only if

$$\int \mathrm{d}V\zeta\eta = \oint \mathrm{d}s\left(b\eta - a\frac{\partial\eta}{\partial n}\right) \tag{6}$$

for any function η harmonic in V, i.e. such that $\nabla^2\eta = 0$ in V.

To prove this proposition first let $\zeta = \nabla^2\psi$ for some function ψ with $\psi\,|_S = a$ and $\partial\psi/\partial n\,|_S = b$. By Green's theorem it results, for any harmonic function η,

$$\begin{aligned}\int \mathrm{d}V\zeta\eta &= \int \mathrm{d}V(\nabla^2\psi)\eta \\ &= \int \mathrm{d}V\psi\nabla^2\eta + \oint \mathrm{d}s\left(\frac{\partial\psi}{\partial n}\eta - \psi\frac{\partial\eta}{\partial n}\right) \\ &= \oint \mathrm{d}s\left(b\eta - a\frac{\partial\eta}{\partial n}\right).\end{aligned}$$

Conversely, let v be a function such that

$$\int \mathrm{d}Vv\eta = \oint \mathrm{d}s\left(b\eta - a\frac{\partial\eta}{\partial n}\right)$$

for any harmonic function η. Then, let ψ be the unique solution of the Poisson equation $\nabla^2\psi = v$ with Dirichlet boundary condition $\psi|_S = a$. By Green's theorem it results, for any harmonic function,

$$\int \mathrm{d}V v\eta = \int \mathrm{d}V(\nabla^2\psi)\eta$$

$$= \int \mathrm{d}V\psi\nabla^2\eta + \oint \mathrm{d}s\left(\frac{\partial\psi}{\partial n}\eta - \psi\frac{\partial\eta}{\partial n}\right)$$

$$= \oint \mathrm{d}s\left(\frac{\partial\psi}{\partial n}\eta - a\frac{\partial\eta}{\partial n}\right).$$

Hence, by the assumption,

$$\oint \mathrm{d}s b\eta = \oint \mathrm{d}s\frac{\partial\psi}{\partial n}\eta.$$

From the arbitrariness of η on S it follows that $\partial\psi/\partial n\,|_S = b$. The use of conditions (6) allows therefore the complete splitting of the biharmonic problem (1)–(3) into the sequence of two second-order problems

$$\nabla^2\zeta = f, \qquad \int \mathrm{d}V\zeta\eta = \oint \mathrm{d}s\left(b\eta - a\frac{\partial\eta}{\partial n}\right), \qquad \forall\eta : \nabla^2\eta = 0, \tag{7}$$

$$\nabla^2\psi = \zeta, \qquad \psi\,|_S = a. \tag{8}$$

While the second Poisson equation is provided with usual Dirichlet boundary condition, the first one is supplemented by conditions of integral type which admit a simple geometrical interpretation. The left hand side of (6) is the scalar product of ζ and η in the Hilbert space L^2 of square integrable functions. When $a = b = 0$, ζ is orthogonal in the L^2 sense to the manifold of the harmonic functions. In the general case of nonhomogeneous boundary conditions for ψ, the projection of ζ on this manifold is determined by a and b.

It is noted that the projection conditions lead to the complete decoupling of the two Poisson equations which result from the biharmonic equation. Therefore, ψ can be calculated directly and no iterative method is required, as in the case of the coupled equation approach[3] to the biharmonic problem.

Furthermore, it is worth pointing out that in equation (8) the Dirichlet condition can be also replaced by the Neumann condition $\partial\psi/\partial n\,|_S = b$, augmented by prescribing $\psi(\mathbf{r}_S) = a(\mathbf{r}_S)$, for a single point $\mathbf{r}_S$ of S. In this case the solvability condition of the Neumann problem

$$\int \mathrm{d}V\zeta = \oint \mathrm{d}s b \tag{9}$$

is automatically satisfied since it is simply the projection condition (6) with respect to the trivial harmonic function $\eta \equiv 1$. Therefore, the use of the integral projection conditions for the first Poisson equation allows a free choice between Dirichlet and Neumann boundary conditions for the second equation.

3. TWO-DIMENSIONAL EQUATIONS

The application of the above remark to the equations of viscous incompressible flows in two dimensions is straightforward. In this case the dimensionless Navier–Stokes equations for the nonprimitive variables, namely, the vorticity ζ and the stream function ψ, are

$$\frac{1}{\mathrm{Re}}\nabla^2\zeta=\frac{\partial\zeta}{\partial t}+(\mathbf{u}\cdot\nabla)\zeta, \tag{10}$$

$$\nabla^2\psi=\zeta, \tag{11}$$

where $\zeta=-\zeta_z$, Re is the Reynolds number, $\mathbf{u}=-\hat{\mathbf{z}}\times\nabla\psi$, $\hat{\mathbf{z}}$ being the unit vector orthogonal to the plane of the motion. The velocity boundary condition $\mathbf{u}|_S=\mathbf{b}(\mathbf{r}_S,t)$, $\mathbf{r}_S\in S$, allows the specification of boundary values for both the stream function and its normal derivative. In fact, incompressibility requires $\oint ds\mathbf{n}\cdot\mathbf{b}=0$ at any time, so that, since the domain V is bounded and simply connected, one can define, apart from an arbitrary additive constant, the single-valued function $a=a(s,t)=\int_{s_0}^{s}ds\mathbf{n}\cdot\mathbf{b}$. Setting $b=b(s,t)=-\boldsymbol{\tau}\cdot\mathbf{b}$, the boundary conditions for ψ read

$$\psi|_S=a, \qquad \partial\psi/\partial n|_S=b. \tag{12}$$

Since the conditions for ζ and ψ discussed in the previous section are encountered also in problem (10)–(12) irrespective of the particular form of the equation for ζ, this problem can be restated in the following form

$$\frac{1}{\mathrm{Re}}\nabla^2\zeta=\frac{\partial\zeta}{\partial t}+(\mathbf{u}\cdot\nabla)\zeta, \tag{13}$$

$$\int dV\zeta\eta=\oint ds\left(b\eta-a\frac{\partial\eta}{\partial n}\right), \qquad \forall\eta:\nabla^2\eta=0; \tag{14}$$

$$\nabla^2\psi=\zeta, \tag{15}$$

$$\psi|_S=a. \tag{16}$$

Therefore, at any time, the projection of the vorticity field ζ on the linear manifold of the harmonic functions is determined by the instantaneous values of the tangential and normal velocity along all the boundary. It is clear that these integral conditions cannot be reduced to conditions of a local type specifying the boundary values of the vorticity.

4. THREE-DIMENSIONAL EQUATIONS

In this section the general case of three-dimensional flows is considered also in view of the growing computational interest in the nonprimitive variable representation of viscous flows in three dimensions.[4–8]. In this case, the explicit separation of the irrotational and rotational components of the velocity field by means of the scalar and vector potentials, respectively, singles out that part of the boundary velocity which actually affects the vorticity field, via the projection conditions for the vorticity vector. Furthermore, it turns out that the projection conditions depend on the gauge which is chosen for the vector potential of the velocity field.

The basic remark given in Section 2 has the following vector analogue in three dimensions:

A divergenceless vector $\boldsymbol{\zeta}$ *in* V is such that $\boldsymbol{\zeta}=\nabla\times\nabla\times\mathbf{A}$, with $\mathbf{n}\times\mathbf{A}|_S=\mathbf{n}\times\mathbf{a}$ and $\mathbf{n}\times\nabla\times\mathbf{A}|_S=\mathbf{n}\times\mathbf{b}$, if and only if

$$\int dV\boldsymbol{\zeta}\cdot\boldsymbol{\eta}=\oint dS(\mathbf{n}\times\mathbf{b}\cdot\boldsymbol{\eta}+\mathbf{n}\times\mathbf{a}\cdot\nabla\times\boldsymbol{\eta}) \tag{17}$$

for any vector field $\boldsymbol{\eta}$ such that $\nabla\times\nabla\times\boldsymbol{\eta}=0$ in V.

Here V is a simply connected domain of the three-dimensional space, $\mathbf{a}$ and $\mathbf{b}$ are vector fields defined on S, the boundary of V, $\mathbf{n}$ is the unit normal vector on S. This can be proven much as in the two-dimensional case. Necessity is a consequency of the vector analogue of

Green's theorem, namely,

$$\int dV[\mathbf{B}\cdot\nabla\times\nabla\times\mathbf{A}-\mathbf{A}\cdot\nabla\times\nabla\times\mathbf{B}]=\oint dS[(\mathbf{n}\times\nabla\times\mathbf{A})\cdot\mathbf{B}+\mathbf{n}\times\mathbf{A}\cdot\nabla\times\mathbf{B}].$$

Sufficiency of condition (17) comes from the fact that the problem

$$\nabla\times\nabla\times\mathbf{A}=\boldsymbol{\zeta},$$

$$\mathbf{n}\times\mathbf{A}\,|_S=\mathbf{n}\times\mathbf{a},$$

has (at least) one solution provided that $\nabla\cdot\boldsymbol{\zeta}=0$ (see, e.g. Morse and Feshbach, *Methods of Theoretical Physics*, 1953, p. 1788) and using again Green's formula. To cope easily[9–10] with general velocity boundary conditions let the scalar potential ϕ and the vector potential $\mathbf{A}$ of the velocity field $\mathbf{u}$ be introduced

$$\mathbf{u}=\nabla\phi+\nabla\times\mathbf{A}. \tag{18}$$

Then the Navier–Stokes equations yield the following system of equations for the vorticity $\boldsymbol{\zeta}=\nabla\times\mathbf{u}$ and the potentials ϕ and $\mathbf{A}$

$$\nabla^2\phi=0, \tag{19}$$

$$\frac{1}{\mathrm{Re}}\nabla^2\boldsymbol{\zeta}=\frac{\partial\boldsymbol{\zeta}}{\partial t}+\nabla\times(\boldsymbol{\zeta}\times\mathbf{u}), \tag{20}$$

$$\nabla\times\nabla\times\mathbf{A}=\boldsymbol{\zeta}. \tag{21}$$

The velocity boundary condition is $\mathbf{u}\,|_S=\mathbf{b}(\mathbf{r}_S,t)$, and the velocity $\mathbf{b}$ prescribed on S is assumed to satisfy at any time

$$\oint dS\mathbf{n}\cdot\mathbf{b}=0. \tag{22}$$

The boundary condition $\mathbf{u}\,|_S=\mathbf{b}$ may be split into

$$\mathbf{n}\cdot(\nabla\phi+\nabla\times\mathbf{A})\,|_S=\mathbf{n}\cdot\mathbf{b}, \tag{23}$$

$$\mathbf{n}\times(\nabla\phi+\nabla\times\mathbf{A})\,|_S=\mathbf{n}\times\mathbf{b}. \tag{24}$$

Furthermore, knowledge of $\mathbf{b}$ over S allows one to specify boundary values of the normal component of $\boldsymbol{\zeta}$ by means of condition

$$\mathbf{n}\cdot\boldsymbol{\zeta}\,|_S=\mathbf{n}\cdot\nabla_\tau\times\mathbf{b}, \tag{25}$$

where ∇_τ is the surface gradient operator (typically, on a stationary rigid body $\mathbf{n}\cdot\nabla_\tau\times\mathbf{b}\equiv 0$).

The coupling of equations (19)–(21) through conditions (23) and (24) is now eliminated, and a system of three equations to be solved in sequence, aside from the coupling due to the nonlinear term, is derived by an argument consisting of three steps: (i) separation of (23) into two independent conditions for ϕ and $\mathbf{A}$; (ii) choice of the homogeneous boundary condition for the tangential components of the vector potential; (iii) use of (17) to derive the integral conditions for the vorticity.

Let the scalar potential be specified to have the boundary condition[10]

$$\mathbf{n}\cdot\nabla\phi\,|_S=\mathbf{n}\cdot\mathbf{b}, \tag{26}$$

which, in view of (22), allows the equation (19) to be solved, apart from an arbitrary additive constant. Due to (23), this implies that

$$\mathbf{n}\cdot\nabla\times\mathbf{A}\,|_S=0, \tag{27}$$

and that (24) can be rewritten in the form

$$\mathbf{n}\times\nabla\times\mathbf{A}\,|_S=\mathbf{n}\times(\mathbf{b}-\nabla\phi\,|_S). \qquad (28)$$

The vector potential $\mathbf{A}$, as defined by (18), is arbitrary by the gradient of any function. Because of (27), i.e. $\mathbf{n}\,.\,\nabla\times\mathbf{A}\,|_S=0$, the tangential projection $\mathbf{A}_\tau$ of $\mathbf{A}$ on S may be expressed as $\mathbf{A}_\tau=\nabla_\tau\chi_\tau$, where χ_τ is a function of the surface co-ordinates.

Let χ be an extension of χ_τ to V (or the solution of the Poisson problem $\nabla^2\chi=\nabla\,.\,\mathbf{A}$, $\chi\,|_S=\chi_\tau$, if $\mathbf{A}$ is requested to satisfy the gauge $\nabla\,.\,\mathbf{A}=0$). Then, the vector field $\mathbf{A}-\nabla\chi$ is a vector potential as $\mathbf{A}$ and, in addition, it has a zero tangential projection on S.[10–11] It follows that the vector potential $\mathbf{A}$ can be chosen so as to satisfy

$$\mathbf{n}\times\mathbf{A}\,|_S=0. \qquad (29)$$

Use of (29) and (28) in relationship (17) directly provides the vectorial form of the projection conditions

$$\int dV\boldsymbol{\zeta}\,.\,\boldsymbol{\eta}=\oint dS\mathbf{n}\times(\mathbf{b}-\nabla\phi)\,.\,\boldsymbol{\eta}, \qquad \forall\boldsymbol{\eta}:\nabla\times\nabla\times\boldsymbol{\eta}=0, \qquad (30)$$

that, together with (25), supplement the vorticity transport equation (20). In its turn, equation (21) is adequately supplemented by boundary condition (29) or (28). The resulting final system for ϕ, $\boldsymbol{\zeta}$ and $\mathbf{A}$, rewritten entirely for clarity, is

$$\nabla^2\phi=0, \qquad (31)$$

$$\mathbf{n}\,.\,\nabla\phi\,|_S=\mathbf{n}\,.\,\mathbf{b}; \qquad (32)$$

$$\frac{1}{\mathrm{Re}}\nabla^2\boldsymbol{\zeta}=\frac{\partial\boldsymbol{\zeta}}{\partial t}+\nabla\times(\boldsymbol{\zeta}\times\mathbf{u}), \qquad (33)$$

$$\int dV\boldsymbol{\zeta}\,.\,\boldsymbol{\eta}=\oint dS\mathbf{n}\times(\mathbf{b}-\nabla\phi)\,.\,\boldsymbol{\eta}, \qquad \forall\boldsymbol{\eta}:\nabla\times\nabla\times\boldsymbol{\eta}=0, \qquad (34)$$

$$\mathbf{n}\,.\,\boldsymbol{\zeta}\,|_S=\mathbf{n}\,.\,\nabla_\tau\times\mathbf{b}; \qquad (35)$$

$$\nabla\times\nabla\times\mathbf{A}=\boldsymbol{\zeta}, \qquad (36)$$

$$\mathbf{n}\times\mathbf{A}\,|_S=0. \qquad (37)$$

The choice of the Euclid invariant gauge $\nabla\,.\,\mathbf{A}=0$ for the vector potential slightly modifies the projection conditions (34) and the problem (36), (37). If $\nabla\,.\,\mathbf{A}=0$, the vector fields $\boldsymbol{\eta}$ can be chosen to satisfy the same gauge, namely, $\nabla\,.\,\boldsymbol{\eta}=0$, and (34) becomes

$$\int dV\boldsymbol{\zeta}\,.\,\boldsymbol{\eta}=\oint dS\mathbf{n}\times(\mathbf{b}-\nabla\phi)\,.\,\boldsymbol{\eta}, \qquad \forall\boldsymbol{\eta}:\nabla^2\boldsymbol{\eta}=0. \qquad (38)$$

On the other hand, if $\nabla\,.\,\mathbf{A}=0$, problem (36), (37) becomes[10]

$$\nabla^2\mathbf{A}=-\boldsymbol{\zeta}, \qquad (39)$$

$$\mathbf{n}\times\mathbf{A}\,|_S=0, \qquad (40)$$

$$\frac{1}{h_1h_2}\frac{\partial}{\partial n}(h_1h_2A_n)\,|_S=0. \qquad (41)$$

Finally, it is worth remembering that, once the system (31)–(37) has been solved, the pressure field p of the considered problem is determined, apart from an arbitrary reference value, by calculating the line integral of the equation

$$\nabla\left(\frac{p}{\rho}+\tfrac{1}{2}|\mathbf{u}|^2+\frac{\partial\phi}{\partial t}\right)=-\frac{\partial\nabla\times\mathbf{A}}{\partial t}-\frac{1}{\mathrm{Re}}\nabla\times\boldsymbol{\zeta}-\boldsymbol{\zeta}\times\mathbf{u}, \qquad (42)$$

where $\mathbf{u}=\nabla\phi+\nabla\times\mathbf{A}$.

5. COMMENTS AND THE AXISYMMETRIC CASE

The direct inspection of (34) shows that the projection conditions depend on the geometrical characteristic of the domain V through the linear manifold of vector fields $\boldsymbol{\eta}$ satisfying $\nabla\times\nabla\times\boldsymbol{\eta}=0$. In three dimensions the vorticity field $\boldsymbol{\zeta}$ has a projection on this manifold which is fixed uniquely by the difference between the tangential component of the velocity prescribed on the boundary and the tangential component of the velocity due to the potential motion at the boundary.

It is interesting to note that, in plane two-dimensional flows, i.e. when $\zeta_x=\zeta_y=A_x=A_y=\eta_x=\eta_y=0$ and only ϕ, ζ_z, A_z, η_z survive, system (31)–(37) becomes a system of three scalar equations which is an alternative and to be preferred to (13)–(16) if one aims at describing separately the inviscid (ϕ) and the viscid (A_z) components of the fluid motion. The equivalence of the two formulations is demonstrated by introducing the harmonic function ϕ' conjugate to ϕ, i.e. satisfying the Cauchy-Riemann conditions $\partial\phi'/\partial x=-\partial\phi/\partial y$, $\partial\phi'/\partial y=\partial\phi/\partial x$, and by observing that the two equations for A_z and ϕ', with the respective boundary conditions, merge into a single Poisson equation for the variable $\psi=A_z+\phi'$. It follows that, in plane problems, the stream function ψ is coincident with the component A_z of the three-dimensional vector potential only when the normal component of the boundary velocity is zero along the entire boundary.

A further point deserves attention about problem (33)–(35). Even when the coupling due to the nonlinear term $\nabla\times(\boldsymbol{\zeta}\times\mathbf{u})$ is disregarded, the presence of the projection conditions prevents the separation of the equations for the components of $\boldsymbol{\zeta}$ also in domains and co-ordinates for which the vector Poisson equation is separable. Axisymmetric flows are an exception to this rule if use is made of the mixed representation of the flow by the nonprimitive variables for the motion in the meridional plane and by the primitive variable for the azimuthal motion. In fact, among the vector fields $\boldsymbol{\eta}$ such that $\nabla\times\nabla\times\boldsymbol{\eta}=0$, one can consider those independent of the azimuthal angle φ and having only the azimuthal component. Considering, for instance, the case of spherical co-ordinates (r,θ,φ), and denoting by $\hat{\boldsymbol{\varphi}}$ the unit vector normal to the meridional plane, we can take $\boldsymbol{\eta}=\hat{\boldsymbol{\varphi}}\eta(r,\theta)$, so that $\nabla\times\nabla\times\boldsymbol{\eta}=0$ becomes $E^2\eta=0$, where

$$E^2=\nabla^2-\frac{1}{r^2\sin^2\theta}, \tag{43}$$

∇^2 being the two-dimensional Laplace operator for the variables in the meridional plane. Then, the azimuthal and meridional components of the velocity and vorticity are introduced according to

$$\mathbf{u}=\hat{\boldsymbol{\varphi}}u+\nabla\times(\hat{\boldsymbol{\varphi}}\psi), \tag{44}$$

$$\boldsymbol{\zeta}=\hat{\boldsymbol{\varphi}}\zeta+\nabla\times(\hat{\boldsymbol{\varphi}}u), \tag{45}$$

where ψ is the stream function variable for the two-dimensional motion in the meridional plane. (Notice that $\psi=A+\phi'$, where A is the azimuthal component of the vector potential $\mathbf{A}$ and ϕ' satisfies $E^2\phi'=0$, $\phi'|_S=a$, the relationship between ϕ' and the scalar potential ϕ being that $\nabla\times(\hat{\boldsymbol{\varphi}}\phi')=\nabla\phi$.)

From (31)–(37) the equations for the unknowns $u(r,\theta)$, $\zeta(r,\theta)$ and $\psi(r,\theta)$ can be obtained in the form

$$\frac{1}{\mathrm{Re}}E^2u=\frac{\partial u}{\partial t}+\hat{\boldsymbol{\varphi}}\,.\,\boldsymbol{\zeta}\times\mathbf{u}, \tag{46}$$

$$u|_S=c; \tag{47}$$

$$\frac{1}{\mathrm{Re}} E^2\zeta = \frac{\partial \zeta}{\partial t} + \hat{\varphi} \,.\, \nabla \times (\boldsymbol{\zeta} \times \mathbf{u}), \tag{48}$$

$$\int \mathrm{d}v r \sin\theta \zeta\eta = -\oint \mathrm{d}s r \sin\theta \left[b\eta - a\left(\frac{\partial \eta}{\partial n} + \frac{\eta}{r\sin\theta}\frac{\partial}{\partial n}(r\sin\theta)\right)\right], \qquad \forall \eta : E^2\eta = 0; \tag{49}$$

$$E^2\psi = -\zeta, \tag{50}$$

$$\psi|_S = a; \tag{51}$$

where

$$a = \left(\int_{s_0}^{s} \mathrm{d}s r \sin\theta \mathbf{n} \,.\, \mathbf{b}\right) \Big/ (r\sin\theta), \tag{52}$$

$$b = -\boldsymbol{\tau} \,.\, \mathbf{b}, \tag{53}$$

$$c = \hat{\varphi} \,.\, \mathbf{b}. \tag{54}$$

The domains of integrals occurring in (49) and (52) belong to a meridional section of the originally axisymmetric three-dimensional domain.

In cylindrical co-ordinates (r, z, φ), the equations for axisymmetric flows have the same form (43)–(54) provided that $\sin\theta$ is replaced by 1.

Finally it is noted that the arguments leading to the projection conditions are completely independent of the form of the dynamical equation governing the evolution of the vorticity field. Therefore these conditions apply to viscous incompressible flows also in the presence of external body forces, such as Coriolis and buoyancy forces.

A further application is to the equations of the magnetohydrodynamics where integral projection conditions for the variables $\nabla \times \mathbf{u}$ and $\nabla \times \mathbf{B}$ must be considered whenever the vector potentials of the velocity $\mathbf{u}$ and of the magnetic field $\mathbf{B}$ are chosen as independent variables.

6. DISCRETIZED PROJECTION CONDITIONS

Discrete analogues of the equations for steady and unsteady flows in two dimensions are now considered focusing on the computational consequences brought about by the projection conditions. A complete error analysis of the numerical schemes to be introduced will not be attempted here, but numerical results of a model problem will be given in the next section.

The discretized version of (14) at time $t^{n+1} \equiv (n+1)\Delta t$ reads

$$\int \mathrm{d}V \zeta^{n+1}\eta_\alpha = \oint \mathrm{d}s\left(b^{n+1}\eta_\alpha - a^{n+1}\frac{\partial \eta_\alpha}{\partial n}\right), \qquad \nabla^2\eta_\alpha = 0, \tag{55}$$

where the integrals and the Laplace operator are to be interpreted in the sense of the assumed spatial discretization obtained by means of finite differences or finite elements.

It can be easily shown that the manifold of the discrete harmonic functions contains exactly as many linearly independent functions as boundary points, say the N discrete harmonic functions η_α, $\alpha = 1, 2, \ldots, N$, vanishing at all N boundary points except one

$$\nabla^2\eta_\alpha = 0, \qquad \eta_\alpha|_S = \delta_{\beta\alpha}, \tag{56}$$

$\delta_{\beta\alpha}$ being the Kronecker symbol. It follows that integral conditions (55) provide the N linearly independent algebraic equations required to close the system of equations resulting from the spatial discretization of the second-order equation for ζ.

The explicit form of conditions (55) depends whether they apply to time-independent or time-dependent equations, and whether explicit or implicit differencing is chosen for the terms of the vorticity transport equation. Computational schemes for steady and unsteady problems are now described which assume the implicit treatment of the linear vorticity–diffusion term $\nabla^2\zeta$ and the explicit treatment of the nonlinear advection term $\mathbf{u}\cdot\nabla\zeta = J(\zeta, \psi)$.

For the time-independent equations the iterative scheme is: start with $\zeta^0 = 0$ and $\psi^0 = 0$; when ζ^m and ψ^m are known, define ζ^{m+1} and ψ^{m+1} as the solutions of

$$\nabla^2\zeta^{m+1} = \mathrm{Re}\, J(\zeta^m, \psi^m), \tag{57}$$

$$\int \mathrm{d}V \zeta^{m+1}\eta_\alpha = \oint \mathrm{d}s\left(b\eta_\alpha - a\frac{\partial\eta_\alpha}{\partial n}\right); \tag{58}$$

$$\nabla^2\psi^{m+1} = \zeta^{m+1}, \tag{59}$$

$$\psi^{m+1}|_S = a. \tag{60}$$

The solution ζ^{m+1} of (57) and (58) is computed as follows. Equation (57) is first solved with arbitrary boundary values in the auxiliary variable ζ_1. Then ζ^{m+1} is sought in the form

$$\zeta^{m+1} = \zeta_1 + \sum_{\alpha=1}^{N} p_\alpha^{m+1}\eta_\alpha. \tag{61}$$

By imposing (58), the vector $\mathbf{p}^{m+1} = \{p_\alpha^{m+1}, \alpha = 1, \ldots, N\}$, is found to be the solution of the linear system

$$A\mathbf{p}^{m+1} = \mathbf{c}^{m+1}, \tag{62}$$

where matrix A and vector $\mathbf{c}^{m+1}$ are defined by

$$A_{\alpha\beta} = \int \mathrm{d}V \eta_\alpha\eta_\beta, \tag{63}$$

$$c_\alpha^{m+1} = -\int \mathrm{d}V \zeta_1\eta_\alpha + \oint \mathrm{d}s\left(b\eta_\alpha - a\frac{\partial\eta_\alpha}{\partial n}\right). \tag{64}$$

(63) shows that A is the matrix of the scalar products among the basis functions η_α (Gram matrix). It follows that A is symmetric definite positive. Vector $\mathbf{c}^{m+1}$ must be calculated at each iteration, whereas matrix A can be generated and factorized once and for all. Furthermore, these constructions depend only on the geometry of the domain, so that solutions at different Reynolds numbers and with different boundary conditions for a fixed domain can be calculated by the same matrix A.

Notice that, for $\mathrm{Re} = 0$, the considered scheme becomes a direct (noniterative) method of solution for the biharmonic equation, dealing with Poisson equations and the symmetric linear system (62) with N unknown, where N is the number of boundary points. The method is such that the calculation of the vorticity at interior points is separated from the calculation of the vorticity at points on the boundary, the projection conditions for the vorticity establishing relationships between vorticity values at all boundary points. An iterative scheme to perform the same kind of separation has been previously proposed by Israeli.[12]

A scheme with the same differencing of (57)–(60) is possible for the time-dependent equations. The algorithm starts with $\zeta^0 = \zeta_0 \equiv \nabla^2\psi_0$ and $\psi^0 = \psi_0$, where ψ_0 is the initial stream function; when the solutions ζ^n and ψ^n at the time $n\Delta t$ are known, define ζ^{n+1} and ψ^{n+1} as the solutions of

$$\left(\nabla^2-\frac{\mathrm{Re}}{\Delta t}I\right)\zeta^{n+1}=-\frac{\mathrm{Re}}{\Delta t}\zeta^n+\mathrm{Re}\,J(\zeta^n,\psi^n), \tag{65}$$

$$\int \mathrm{d}V\zeta^{n+1}\eta_\alpha=\oint \mathrm{d}s\left(b^{n+1}\eta_\alpha-a^{n+1}\frac{\partial\eta_\alpha}{\partial n}\right); \tag{66}$$

$$\nabla^2\psi^{n+1}=\zeta^{n+1}, \tag{67}$$

$$\psi^{n+1}|_S=a^{n+1}; \tag{68}$$

where I denotes the identity matrix for finite differences whereas the consistent mass matrix for finite elements.

This time the solution ζ^{n+1} of (65) and (66) is sought in the form

$$\zeta^{n+1}=\zeta_1+\sum_{\alpha=1}^{N}p_\alpha^{n+1}\theta_\alpha, \tag{69}$$

where ζ_1 satisfies (65) with arbitrary boundary conditions, and the functions θ_α, $\alpha=1,2,\ldots,N$, satisfy the Helmholtz equations

$$\left(\nabla^2-\frac{\mathrm{Re}}{\Delta t}I\right)\theta_\alpha=0,\qquad \theta_\alpha|_S=\delta_{\beta\alpha}. \tag{70}$$

By imposing (66), one finds the following linear system for the unknown vector $\mathbf{p}^{n+1}$

$$B\mathbf{p}^{n+1}=\mathbf{c}^{n+1}, \tag{71}$$

where

$$B_{\alpha\beta}=\int \mathrm{d}V\theta_\alpha\eta_\beta, \tag{72}$$

$$c_\alpha^{n+1}=-\int \mathrm{d}V\zeta_1\eta_\alpha+\oint \mathrm{d}s\left(b^{n+1}\eta_\alpha-a^{n+1}\frac{\partial\eta_\alpha}{\partial n}\right). \tag{73}$$

By means of the discretized version of Green's theorem it can be shown from (56) and (70) that matrix B of the time-dependent equations is also symmetric.

The occurrence of Poisson and Helmholtz problems in (56), (57), (59) and (65), (70) makes the above schemes interesting particularly in situations where direct algorithms for the solution of second-order equations are available. Anyway, these schemes, which are based on and impose the exact constraints for the vorticity, provide at least a rationale for deriving approximate relationships of increasing accuracy for the vorticity boundary values.

7. NUMERICAL EXAMPLE

In this section the numerical schemes (57)–(64) and (65)–(73) are employed to calculate the viscous flow in a two-dimensional square cavity, bounded by three stationary walls and by a fourth upper wall sliding at a constant speed: the driven cavity problem (see, for instance, References 13–16).

Numerical results are reported here to serve as a simple illustration of the use, and to give some valuation of convergence and efficiency, of schemes which employ the projection conditions. No extensive investigation of the above properties and no detailed comparison with other methods is attempted here.

For the considered problem all the variables are made dimensionless by taking the depth L of the square cavity and the velocity U of the upper wall as the units. A rectangular co-ordinate system (x, y) with its origin at the lower left corner of the cavity is introduced so that the boundary conditions to be satisfied are

$$\begin{aligned} &\psi = \partial\psi/\partial x = 0 && \text{at } x = 0 \text{ and } x = 1,\\ &\psi = \partial\psi/\partial y = 0 && \text{at } y = 0,\\ &\psi = 0,\ \partial\psi/\partial y = -1 && \text{at } y = 1. \end{aligned} \tag{74}$$

Finite differences are employed over the uniform mesh

$$x_i = ih,\ y_j = jh, \qquad i, j = 0, 1, \ldots, r,$$

where $h = 1/r$. The five-point approximation of ∇^2 is used and the discrete integration over V is performed with weighting coefficients 1, 1/2 and 1/4 for interior, boundary and corner points, respectively. The choice $r = 2^k$ allows the direct solution of the involved Poisson and Helmholtz equations by means of the fast Poisson solvers. Computations have been performed on an IBM 370/165 using single precision arithmetic, except for accumulation of scalar products and of linear combinations (61) and (69) where double precision has been used.

As shown by Israeli,[12] the iterative solution of the vorticity/stream function equations is numerically unstable, even in the linear case $\mathrm{Re} = 0$, when the two Poisson equations are solved exactly at each iteration and the vorticity boundary values are calculated in terms of stream-function values without any averaging or relaxation process on the boundary. Actually, in the test problem considered here if the commonly used approximation of boundary vorticity

$$\zeta_S^{m+1} = 2(\psi_{S\pm1}^m - bh)/h^2 \tag{75}$$

is employed, numerical divergence occurs at $\mathrm{Re} = 0$ after 210, 68, 43 and 29 iterations on the mesh $r = 4, 8, 16$ and 32, respectively. On the contrary at $\mathrm{Re} = 0$ the steady algorithm (57)–(64) provides the solution directly on any mesh. The fields ζ and ψ obtained by this method in the case $r = 32$ are shown in Figure 1.

At $\mathrm{Re} > 0$ the number of iterations required to obtain the solution within a fixed accuracy depends on the finite difference approximation of the nonlinear term $J(\zeta, \psi)$. This dependence is not important when $\mathrm{Re} \leqslant 10$ and solutions are obtained, with a maximum local error between two subsequent iterates of ζ and ψ less than 5.10^{-6}, in less than 10 iterations irrespective of the discrete approximation chosen for J and of the considered mesh. At low Reynolds numbers the convergence rate is independent of the mesh size and is similar to the one occurring in the iterative solution of the nonlinear fourth-order equation by means of the direct biharmonic solvers.[17]

At $\mathrm{Re} \sim 100$ the explicit treatment of the nonlinear term in the steady algorithm is such that the convergence rate depends on the spatial discretization chosen for the Jacobian.[18] The differencing which shows the best convergence is the centred diagonal differencing of J written in divergence form. The solution for $\mathrm{Re} = 100$ and $r = 32$, starting from $\zeta^{m=0} = \psi^{m=0} = 0$, is obtained in 176 iterations with a maximum local error $< 5 \times 10^{-6}$. This solution is shown in Figure 2. Its computation has required a time of less than 10 minutes including generation and factorization of matrix A. The computed solutions shown in Figures 1 and 2 are in fair accordance with results previously obtained by different computational schemes.[13–15] For instance, the positions of the main vortex centre are coincident, within plot

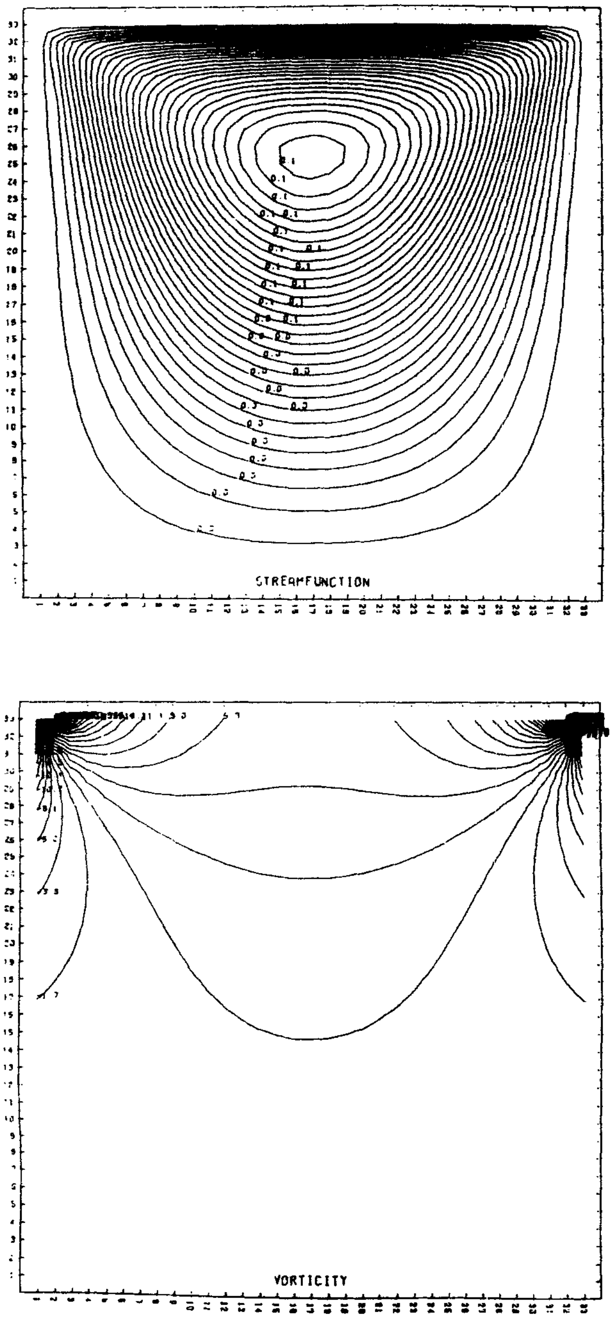
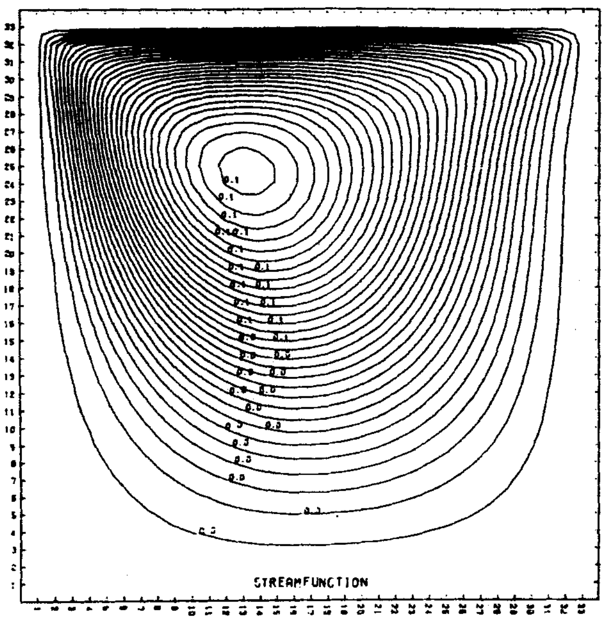


Figure 1. Vorticity and stream function patterns obtained by direct method for the steady flow in a square cavity at Re = 0



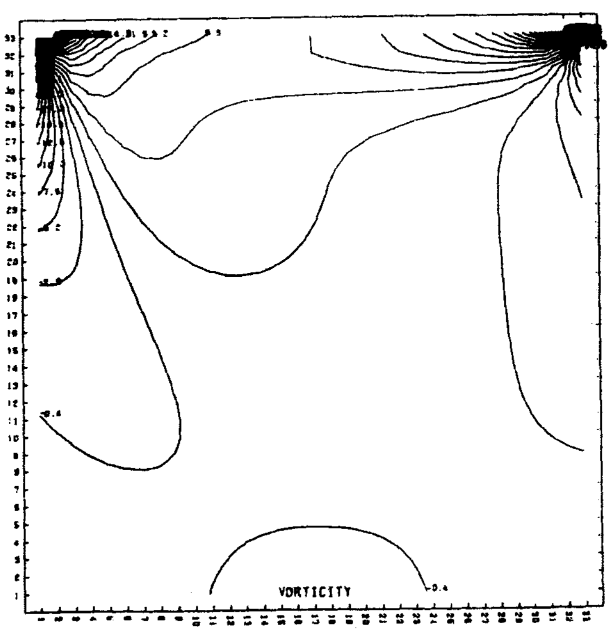


Figure 2. Vorticity and stream function patterns for the steady flow in a square cavity at Re = 100

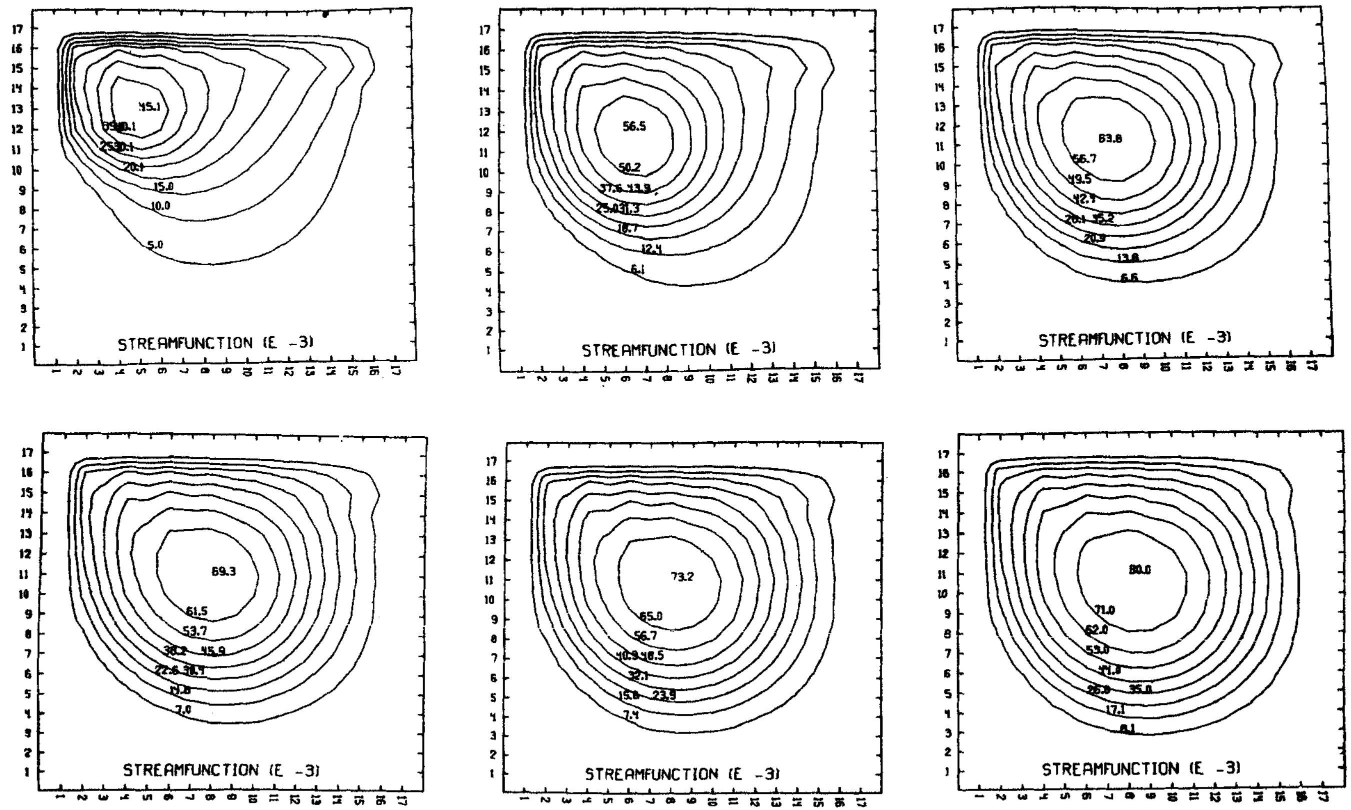


Figure 3. Stream function patterns for the unsteady flow in a square cavity with impulsive start of the driving wall. Re = 1000, $\Delta t = 0{\cdot}05$, $t = 5, 10, 15, 20, 25, 50$

accuracy, with those in Burggraf's paper. At these points $\psi = 0{\cdot}0995$, $\zeta = 3{\cdot}21$ at Re = 0, and $\psi = 0{\cdot}101$, $\zeta = 3{\cdot}16$ at Re = 100, quite close to the reference values of (0·0998, 3·20) and (0·101, 3.14), respectively, obtained by Burggraf using a uniform mesh of 41×41 points. The two secondary eddies at the lower corners reported by several authors are also present in our numerical solutions although not shown in the figures for plotting convenience.

Finally we have calculated by means of the unsteady algorithms (65)–(73) the evolution of the flow field in the cavity when the upper wall is started impulsively with the fluid at rest initially. We have considered the case Re = 1000, $\Delta t = 0{\cdot}05$, $r = 16$ using the fourth-order centred-difference approximation for the derivatives in the advection term, as suggested by Ozawa.[16] In Figure 3 we report the stream function at times $t = 5$, 10, 15, 20, 25 and 50. At $t = 50$ the maximum local differences between (ζ^n, ψ^n), and $(\zeta^{n+1}, \psi^{n+1})$ is $6{\cdot}2 \times 10^{-4}$. The vortex centre is located at $x = 0{\cdot}44$, $y = 0{\cdot}59$ where $\zeta = 1{\cdot}522$ and $\psi = 0{\cdot}080$, in fair agreement with numerical results $x = 0{\cdot}453$, $y = 0{\cdot}587$, $\zeta = 1{\cdot}458$ and $\psi = 0{\cdot}0756$ obtained by Ozawa.

8. CONCLUSION

Although the idea of conditions of an integral type for the vorticity has been anticipated by Cheng,[19] their explicit form and their geometrical interpretation does not seem to have been fully recognized. These conditions provide the vorticity transport equation with an independent conditioning. In fully three-dimensional problems, the vector projection conditions do couple the three vectorial components of the vorticity. On the other hand, in two-dimensional and axisymmetric problems, the projection conditions are of a scalar character and involve one component of the vorticity vector only.

The integral conditions have been implemented into a few computational schemes for the solution of the steady and unsteady two-dimensional equations in discretized form. The numerical results obtained in finite difference calculations of a model problem indicate that the use of the projection conditions avoids the numerical instability that is encountered when vorticity boundary values are specified in terms of stream function values without a relaxation process on the boundary. The method has been successfully tested up to Reynolds numbers of 1000. In the authors opinion, its simplicity and logical coherence seem to compensate for a somewhat larger computational effort.

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